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FEEDBACK DECOMPOSITION OF NONLINEAR CONTROL SYSTEMS

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Feedback decomposition of nonlinear control systems*)

by

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ABSTRACT

By using the recently developed (differential) geometric approach to nonlinear systems a feedback decomposition for nonlinear control systems is derived.

KEY WORDS & PHRASES: *nonlinear control systems; differential geometric methods; controllability distributions; parallel decomposition*

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*) This report will be submitted for publication elsewhere.

1. Introduction

Consider a control system of the form

$$\text{(1.1a)} \qquad \dot{x} = A(x) + \textstyle\sum_{i=1}^{m} B_i(x)u_i$$

$$\text{(1.1b)} \qquad z_i = H_i(x) \qquad , \quad i = 1,\dots,m$$

where x are local coordinates of a smooth n-dimensional manifold M, $A, B_1, \dots, B_m$ are smooth vector fields on M and $H_i : M \to N_i$ is a smooth output map from M to a smooth p_i-($p_i \geq 1$) dimensional manifold N_i for $i = 1,\dots,m$. We assume that each H_i, $i = 1,\dots,m$, is a surjective submersion. Furthermore we will assume that the system (1.1a) is strongly accessible (see [12]).

In this note we will study the *static state feedback noninteracting control problem*. That is, see [4], we seek a control law of the form

$$\text{(1.2)} \qquad u = \alpha(x) + \beta(x)v$$

where $\alpha : M \to \mathbb{R}^m$, $\beta : M \to \mathbb{R}^{m\times m}$ are smooth maps, $\beta(x) = (\beta_{ij}(x))$ is nonsingular for all x in M and $v = (v_1,\dots,v_m)^t \in \mathbb{R}^m$. Let $\tilde{A}(x) = A(x) + \sum_{i=1}^{m} B_i(x)\alpha_i(x)$ and $\tilde{B}_i(x) = \sum_{j=1}^{m} B_j(x)\beta_{ji}(x)$. Then in suitable local coordinates the modified dynamics $\dot{x} = \tilde{A}(x) + \sum_{i=a}^{m} \tilde{B}_i(x)v_i$ should read

$$\text{(1.3a)} \qquad \begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_m \end{pmatrix} = \begin{pmatrix} \tilde{A}_1(x_1) \\ \tilde{A}_2(x_2) \\ \vdots \\ \tilde{A}_m(x_m) \end{pmatrix} + \begin{pmatrix} \tilde{B}_1(x_1) & & \bigcirc \\ & \tilde{B}_2(x_2) & \\ & & \ddots \\ \bigcirc & & \tilde{B}_m(x_m) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \end{cases}$$

$$\text{(1.3b)} \qquad \begin{cases} z_1 = H_1(x_1) \\ \quad\vdots \\ z_m = H_m(x_m) \end{cases}$$

where $x = (x_1,\dots,x_m)$ with each x_i and z_i being possibly a vector. For linear systems the above problem - the Restricted Decoupling Problem (RDP) - has been solved under the additional assumption that the set of outputs is "complete", i.e. $\bigcap_{i=1}^{m} \operatorname{Ker} D_i = 0$, see [13]. In the solution we present here we use as key tools the so called (regular) controllability distributions, introduced in [8]. In this way our approach completely fits in the systematic work on the generalization of Wonham's geometric approach to linear systems, see e.g. [3-10]. We note that a *parallel decomposition* as in (1.3a) has been studied in [11]. We also note that similar results are derived in [4] and, in a different style in [1]. The main purpose of this note is to show that the solution of the nonlinear RDP also can be derived by directly generalizing the theory of [13].

2. Problem formulation

Recall the following definitions, see [3-9].

DEFINITION 2.1. An involutive distribution D of fixed dimension, on M, is *controlled invariant* for the system (1.1a) if there exists a feedback of the form (1.2) such that the modified dynamics $\dot{x} = \tilde{A}(x) + \sum_{i=1}^{m} \tilde{B}_i(x)v_i$ leaves D invariant, i.e.

$$[\tilde{A},D] \subset D$$

$$[\tilde{B}_i,D] \subset D, \quad i = 1,\dots,m.$$

DEFINITION 2.2. An involutive distribution D of fixed dimension, on M, is a *regular controllability distribution* of the system (1.1a) if it is controlled invariant for the system and moreover

$$D = \text{involutive closure of } \{ad_{\tilde{A}}^k \tilde{B}_i \mid k \in \mathbb{N}, i \in I\}$$

for a certain subset $I \subset \{1,\dots,m\}$.

Instead of the above notion of controlled invariance it is sufficient to use a somewhat weaker concept.

DEFINITION 2.3. An involutive distribution D of ficed dimension, on M, is *locally controlled invariant* for the system (1.1a) if locally around each point $x_0 \in M$ there exists a feedback of the form (1.2) such that the modified dynamics $\dot{x} = \tilde{A}(x) + \sum_{i=1}^{m} \tilde{B}_i(x)v_i$ leaves D invariant.

Similarly one defines a local version of definition 2.2: the regular local controllability distributions.

In considering the static state feedback noninteracting control problem we seek regular local controllability distributions $R_1,\dots,R_m$ defined by

$$(2.1) \qquad R_i := \text{involutive closure of } \{ad_{\tilde{A}}^k \tilde{B}_i \mid k \in \mathbb{N}\}$$

where $\tilde{A}$ and $\tilde{B}_i$ are as in (1.3a), $i = 1,\dots,m$.

REMARK: In the local coordinates of (1.3a) we see that $R_i = \text{Span}\{\frac{\partial}{\partial x_i}\}$, and clearly each distribution R_i satisfies $[\tilde{A},R_i] \subset R_i$ and $[\tilde{B}_j,R_i] \subset R_i$, $j = 1,\dots,m$, $i = 1,\dots,m$.

Assuming (2.1) we see that

(2.2) $\quad R_i \subset \mathrm{Ker}\, H_{j*} =: K_j \quad j \neq i,\ i,j = 1,\ldots,m,$

which exactly means that $v_j(\cdot)$ does not affect the output $z_i(\cdot)$, for $j \neq i$. Secondly we have the nonlinear version of *output controllability*, that is

(2.3) $\quad H_{i*}(R_i) = TN_i \qquad\qquad i = 1,\ldots,m.$

This follows from the fact that the system (1.1a) is strongly accessible, so also (1.3a) is strongly accessible. But then each of the systems $\dot{x}_i = \tilde{A}_i(x_i) + \tilde{B}_i(x_i)v_i$ is strongly accessible and by the fact that the map H_i is a surjective submersion we see that the set of reachable output values has nonempty interior in N_i for all $i = 1,\ldots,m$.
Thus the static state feedback noninteracting control problem can be stated as follows.
Given the system (1.1a,b) *find* (*if possible*) *a local feedback law of the form* (1.2) *such that* (2.2) *and* (2.3) *hold for the distributions* R_i *defined by* (2.1)
Now, as in the linear case, there is a compatibility problem (see [13]). Clearly if we have controlled invariant distributions $D_1,\ldots,D_m$, then by no means it follows that there exists a local feedback (1.2) which leaves each of them invariant. Therefore we make the following assumption

(2.4) $\quad \bigcap_{i=1}^{m} \mathrm{Ker}\, H_{i*} = 0,$

which means that the map

$$H : M \to N_1 \oplus N_2 \oplus \ldots \oplus N_m, \quad H(x) = (H_1(x),\ldots,H_m(x))$$

is locally injective.

3. Main theorem

Define $\quad R_i^* :=$ supremal regular local controllability distribution in $\bigcap_{j\neq i} \mathrm{Ker}\, H_{j*}$, $i = 1,\ldots,m$.

REMARK: R_i^* is well defined, see [6,8] but probably the dimension is not fixed.

THEOREM 3.1. *Under the assumption* (2.4) *and the assumption that each* R_i^* *has fixed dimension*, $i = 1,\ldots,m$, *the static state feedback noninteracting control problem is solvable in a local fashion if and only if*

(3.1) $\quad R_i^* + K_i = TM.$

PROOF: Assume (3.1) holds, then (2.2) and (2.3) are true for R_i^*. We show next that the $\hat{K}_i := \bigcap_{j\neq i} \text{Ker } H_{j^*}$, $i = 1,\dots,m$, are independent. Indeed

$$\hat{K}_i \cap \sum_{j\neq i} \hat{K}_j = \Big(\bigcap_{r\neq i} \text{Ker } H_{r^*}\Big) \cap \sum_{j\neq i} \Big(\bigcap_{s\neq j} \text{Ker } H_{s^*}\Big)$$

$$\subset \Big(\bigcap_{r\neq i} \text{Ker } H_{r^*}\Big) \cap \text{Ker } H_{i^*} = \bigcap_{r=1}^{m} \text{Ker } H_{r^*} = 0.$$

Since $R_i^* \subset \hat{K}_i$, $i = 1,\dots,m$, it follows that the R_i^* are independent. In the next step we will show that the R_i^* are compatible, i.e. there is a local feedback (1.2) which leaves each of the distributions R_i^* invariant. From (3.1) we see that for each $i = 1,\dots,m$ $R_i^* \neq 0$. For if $R_i^* = 0$ for an $i \in \{1,\dots,m\}$, then $K_i = TM$, which means that $z_i = D_i(x)$ is constant. Therefore we know, by the independence of the R_i^* that locally there exist independent vector fields $0 \neq \bar{B}_i$ with $\bar{B}_i \in R_i^* \cap \text{Span}\{B_1,\dots,B_m\}$, $i = 1,\dots,m$. So $\text{Span}\{B_1,\dots,B_m\} = \text{Span}\{\bar{B}_1,\dots,\bar{B}_m\}$. We also have that $\dim R_i^* \geq p_i$ (by assumption R_i^* has fixed dimension) and thus from the independency of the R_i^* we have $R_1^* = \dots + R_m^* = TM$. Thus the distributions $R_1^*,\dots,R_m^*$ are *simultaneously integrable* (see definition 3.1 and lemma 3.1 of [11]). So locally around each point $x_0 \in M$ there exist coordinates such that $R_i^* = \text{Span}\{\frac{\partial}{\partial x_i}\}$ $i = 1,\dots,m$, with each x_i possibly being a vector. Now from the fact that the distributions R_i^* are locally controlled invariant we have that

(3.2a) $\qquad [\bar{B}_i, R_j^*] \subset R_j^* + \text{Span}\{\bar{B}_1,\dots,\bar{B}_m\}$, $i = 1,\dots,m$

(3.2a) $\qquad [A, R_j^*] \subset R_j^* + \text{Span}\{\bar{B}_1,\dots,\bar{B}_m\}$

for all $j = 1,\dots,m$.
From (3.2a) we see that

(3.3) $$[\bar{B}_1, R_2^* + \dots + R_m^*] \subset R_2^* + \dots + R_m^* + \text{Span}\{\bar{B}_1,\dots,\bar{B}_m\}$$

$$= R_2^* + \dots + R_m^* + \text{Span}\{\bar{B}_1\},$$

where the last equality follows from the fact that $\bar{B}_i \in R_i^*$, $i = 1,\dots,m$. Note also that the distribution $R_2^* + \dots + R_m^*$ is involutive, cf. [11]. Now from (3.3) and [5,7] it follows that there locally exists a vector field $\tilde{B}_1$ such that $\text{Span}\{\tilde{B}_1\} = \text{Span}\{\bar{B}_1\}$ and $[\tilde{B}_1, R_2^* + \dots + R_m^*] \subset R_2^* + \dots + R_m^*$. Therefore in the coordinate system constructed above we have that $\tilde{B}_1(x) = (\tilde{B}_1(x_1), 0,\dots,0)^t$.
Similarly we construct vector fields $\tilde{B}_i$, $i = 2,\dots,m$, such that $[\tilde{B}_i, R_1^* + \dots + R_{i-1}^* + R_{i+1}^* + \dots + R_m^*] \subset R_1^* + \dots + R_{i-1}^* + R_{i+1}^* + \dots + R_m^*$ and $\text{Span}\{\tilde{B}_i\} = \text{Span}\{\bar{B}_i\}$.
Thus

$$B_i(x) = (0,\dots,0, B_i(x_i), 0,\dots,0)^t.$$

Next from (3.2b) we see that

$$(3.4) \qquad [A, R_2^* + \dots + R_m^*] \subset R_2^* + \dots + R_m^* + \mathrm{Span}\{\bar{B}_1\}$$

and therefore we can construct a local feedback $u = \bar{B}(x)\alpha_1(x)$ such that $\tilde{A}(x) = A(x) + \bar{B}_1(x)\alpha_1(x)$ satisfies (cf. [3]) $[\tilde{A}, R_2^* + \dots + R_m^*] \subset R_2^* + \dots + R_m^*$. Similarly for the distribution $R_1^* + \dots + R_{i-1}^* + R_{i+1}^* + \dots + R_m^*$ we construct a feedback $u = \bar{B}_i(x)\alpha_i(x)$ such that the modified dynamics leave this distribution invariant. Finally by applying the total feedback $u = \bar{B}_1(x)\alpha_1(x) + \dots + \bar{B}_m(x)\alpha_m(x)$ we obtain that $A(x) = (A_1(x_1), A_2(x_2), \dots, A_m(x_m))$. So we have established a local feedback (1.2) such that the modified dynamics are as in (1.3a) and also from (3.1) (1.3b) is satisfied. Furthermore we note that each system $\dot{x}_i = A_i(x_i) + B_i(x_i)v_i$ is strongly accessible and we have that

$$R_i^* = \text{involutive closure of } \{ad_{\tilde{A}}^k \tilde{B}_i \mid k \in \mathbb{N}\},\ i = 1,\dots,m.$$

Conversely from the fact that the R_i^* are supremal relative to the condition (2.2) and from (2.3) - which is equivalent to $R_i + K_i = TM$ - it follows that (3.1) is necessary. □

4. Remarks

(i) In lemma 3.1 of [11] the distributions $D_1,\dots,D_L$ should be independent, i.e. for each disjoint subset I_1 and I_2 of $\{1,\dots,L\}$ one has that $D^{I_1} \cap D^{I_2} = \underline{0}$.

(ii) $[ad_{\tilde{A}}^k \tilde{B}_i, ad_{\tilde{A}}^\ell \tilde{B}_j] = 0$ for all $k,\ell \in \mathbb{N}$ and $i \neq j$, (see also [11]).

(iii) If the number of output channels is smaller than the number of inputs the above procedure still works in a slightly modified way. Namely there are more than one independent vectorfields $\bar{B}_i$ in $R_i^* \cap \mathrm{Span}\{B_1,\dots,B_m\}$ and/or there exist some additional input vector fields $\tilde{B}_k$ which do not belong to one of the distributions R_i^*, but - after applying feedback - also have the form $\tilde{B}_k(x) = (\tilde{B}_k^1(x_1), \tilde{B}_k^1(x_2), \dots, \tilde{B}_k^m(x_m))^t$. These vector fields are superfluous for the whole control synthesis of the system.

(iv) Each of the systems $\dot{x}_i = \tilde{A}_i(x_i) + \tilde{B}_i(x_i)v_i$, $z_i = H_i(x_i)$ is strongly invertible, see [2]. This has also been clarified in a geometric way in [9], and follows directly from the condition that $R_i^* + K_i = TM$, so R_i^* is not contained in $\mathrm{Ker}\, H_{i*}$. We also note that the situation described in theorem 3.1 is even more special. Namely the system $\dot{x}_i = \tilde{A}_i(x_i) + \tilde{B}_i(x_i)v_i$ is strongly invertible with respect to each of the components of the output z_i.

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